THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 50, No. 8, October, 1971
Printed in U.S.A.

Improved Intersymbol Interference Error Bounds in Digital Systems

By Y. S. YEH and E. Y. HO

(Manuscript received April 21, 1971)

A thorough solution to the problem of determining the error rate of a digital communication system with intersymbol interference and additive Gaussian noise is presented in this paper. The solution achieves for the first time a combination of computational simplicity and a high degree of accuracy, and is obtained by deriving tight upper and lower bounds on the error rate. It is shown that, for a system with a normalized peak distortion less than unity, these bounds can be made to differ by an arbitrarily small amount. The numerical evaluation of the bounds takes less than one second on the GE-Mark II time-sharing system for almost all the cases.

Examples are given for 2M-ary digital systems to demonstrate the accuracy and computational efficiency of our method. The results show that our estimates of error rate are generally orders of magnitude better than the Chernoff bound. For example, in the case of an ideal bandlimited system [(sin t)/t pulse shape] with a signal-to-noise ratio of 16 dB and a sampling instant deviation of 0.05 from the optimum value, the lower and upper bounds on the error rate are 1.1×10^{-8} *and* 1.2×10^{-8}, *respectively.*

This method can also be applied to the calculation of the performance of certain phase-shift-keyed systems and certain systems with co-channel interference.

I. INTRODUCTION

In many cases the transmission efficiency of a digital system is largely limited by intersymbol interference rather than by additive noise. Intersymbol interference may result from imperfect design of the filters, distortion in the transmission channel, nonideal sampling instant, or nonideal demodulating carrier phase. In analyzing such a digital data system, it is important to determine the system error rate due to intersymbol interference and additive noise.

Various methods[1-7] to evaluate the error rate have been proposed. They provide either a loose upper bound of the error rate or the error rate of a channel with truncated impulse response.

In this paper we present a simple method to evaluate both an upper and a lower bound of the error rate without invoking the finite pulse-train approximation. Furthermore, it is shown that for a system with a normalized peak distortion less than unity, the upper and lower bounds can be made arbitrarily close thus obtaining an accurate estimate of the error rate of the system. This method can be applied to $2M$-ary AM and coherent phase-shift-keyed systems.

The data system model will be described briefly in Section II. Various proposed techniques to evaluate the error probability and their drawbacks are discussed in Section III. In Section IV, we will present new upper and lower bounds and the computation of the bounds by a series expansion. Applications and the convergence properties of the bounds are described in Section V. Throughout, additive Gaussian noise and independence of information digits are assumed.

II. BRIEF DESCRIPTION OF THE SYSTEM

A simplified block diagram of a digital AM data system is shown in Fig. 1. We assume that an impulse $\delta(t)$ having amplitude a_l is transmitted through the channel every T seconds. The system transfer function is

$$R(\omega) = S(\omega)T(\omega)E(\omega). \tag{1}$$

In the absence of channel noise, a sequence of input signals,

$$\sum_{l=-\infty}^{\infty} a_l \, \delta(t - lT), \tag{2}$$

will generate a corresponding output sequence,

$$\sum_{l=-\infty}^{\infty} a_l r(t - lT), \tag{3}$$

where $r(t)$ is the Fourier transform of $R(\omega)$, $\{a_l\}$ is a sequence of independent random variables, and $a_l = \pm 1, \pm 3, \cdots \pm (2M - 1)$ with equal probability for all integers, l. We also assume that additive Gaussian noise is present in the system. Thus the corrupted received sequence at the input to the receiver detector is

$$y(t) = \sum_{l=-\infty}^{\infty} a_l r(t - lT) + n(t), \tag{4}$$

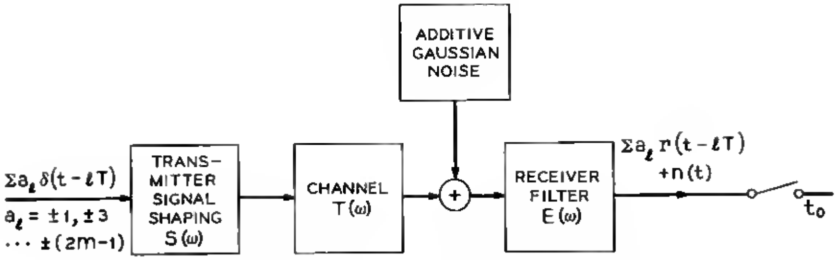

Fig. 1—Simplified block diagram of a $2M$-ary data system.

where $n(t)$ is additive Gaussian noise with power σ^2 watts. At the detector, $y(t)$ is sampled every T seconds to determine the amplitude of the transmitted signal. At sampling time t_0, the sampled signal is

$$y(t_0) = a_0 r(t_0) + \sum_{\substack{l=-\infty \\ l \neq 0}}^{\infty} a_l r(t_0 - lT) + n(t_0). \tag{5}$$

The first term is the desired signal while the second and the third terms represent the intersymbol interference and the Gaussian noise respectively.

The set of slicing levels is[1]

$$0, \ \pm 2r(t_0), \ \pm 4r(t_0), \ \cdots, \ \pm(2m-2)r(t_0). \tag{6}$$

Based on the decision levels given by equation (6), for a particular transmitted signal level, a_0, the conditional error probability is

$$P_r(e/a_0) = \begin{cases} P\{y(t_0) \geqq -2(m-1)r(t_0)\}, & a_0 = -(2m-1) \\ P\{y(t_0) \leqq 2(m-1)r(t_0)\}, & a_0 = 2m-1 \\ P\{[y(t_0) \geqq (a_0+1)r(t_0)]U[y(t_0) \leqq (a_0-1)r(t_0)]\}, & \\ & a_0 \neq \pm(2m-1), \end{cases} \tag{7}$$

where AUB is the union of the events A and B.

Substituting equation (5) into (7), we obtain

$$P_r(e/a_0) = \begin{cases} P\{\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}, & a_0 = -(2m-1) \\ P\{\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \leqq -r(t_0)\}, & a_0 = 2m-1 \\ P\{[\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)]U & \\ \cdot[\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \leqq -r(t_0)]\}, & a_0 \neq \pm(2m-1). \end{cases} \tag{8}$$

Since $\sum_{l \neq 0} a_l r(t_0 - lT)$ and $n(t_0)$ are equally likely to be positive or negative, equation (8) reduces to

$$P_r(e/a_0) = \begin{cases} P\{\sum\limits_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}, & a_0 = \pm(2m - 1) \\ 2P\{\sum\limits_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}, & a_0 \neq \pm(2m - 1). \end{cases} \tag{9}$$

The error rate of the system is

$$\begin{aligned} P_e &= \sum_{\text{all } a_0} P_r(e/a_0) P_r(a_0) \\ &= [(2m - 1)/m] P\{\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}. \end{aligned} \tag{10}$$

We notice that in equation (10) the variables m, a_l , and $n(t_0)$ have already been defined. The sequence $r(t_0 - lT)$ is assumed to be known[*] in the following sense:

$$r(t_0 - lT) \quad \text{is finite and known} \quad \forall\, l \in S_N , \tag{11}$$

where S_N is a set of $N + 1$ distinct integers (including $l = 0$) and[†]

$$\sum_{l \notin S_N} r^2(t_0 - lT) = \sigma_r^2 < \infty . \tag{12}$$

Define

$$X = \sum_{l \neq 0} a_l r(t_0 - lT). \tag{13}$$

From equation (12) we conclude that the infinite sum X converges absolutely to a random variable and equation (10) can be alternately written as

$$P_e \int_{\text{all } X} (2\pi\sigma^2)^{-1} \int_{-\infty}^{0} \exp\{-[y - r(t_0) + X]^2/2\sigma^2\}\, dy\, dF(X). \tag{14}$$

III. REVIEW OF EXISTING METHODS

The existing methods of evaluating equation (10) can be divided into the following categories.

3.1 *Worst Case Estimate*

A worst case sequence[1] or "eye pattern" analysis is frequently used to analyze a data system. The error probability is estimated by setting

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* The sequence $r(t_0 - lT)$ is either experimentally determined or calculated through the system transfer function.

† σ_r^2 is obtained through the application of Parseval's theorem to equivalent Nyquist pulse (p. 47, Ref. 1).

$\sum_{l \neq 0} a_l r(t_0 - lT)$ to its worst case value in equation (10). In many cases, this estimate is exceedingly pessimistic since the occurrence of such a worst case sequence is extremely rare.

3.2. *Chernoff Bound*

Recently, Saltzberg[2] and Lugannani[3] applied the Chebyshev inequality to equation (10) to obtain the upper bound on error probability. We have shown in Ref. 6 that these upper bounds are in many cases still too pessimistic by orders of magnitude.

3.3. *Finite Truncated Pulse Train Approximation*[4,5]

When $r(t)$ decreases rapidly relative to the sampling period T, we may approximate the channel by a finitely truncated pulse train. The error rate can be calculated by enumerating all the possible combinations of intersymbol interference. However, since each calculation of the conditional error probability takes a great deal of computer time, the number of m^N must be held to several thousand.[1] This limitation leads to a poor approximation of the true channel, and the error probability so obtained is not very useful. Recently, Hill[8] has reported that by computer simulation of the density function of X_N, the computation time can be reduced.

3.4 *Series Expansion Method*

Recently, Ho and Yeh[6] and, independently, Shimbo and Celebiler[7] discovered that equation (10) can be calculated in terms of an absolutely convergent series involving moments of the intersymbol interference.* Furthermore, the moments can be obtained readily through recurrence relations, and the computation time is greatly reduced. A better approximation of the real channel can be obtained by increasing the number of terms in the pulse train approximation. However, the error in the P_e estimate introduced by the truncation of the system impulse response is still unknown.

IV. ERROR BOUNDS AND COMPUTATION TECHNIQUES

In this section we shall derive new upper and lower bounds on the error rates and define the range of applicability of our method. No truncation of the intersymbol interference is required. Furthermore, this method will give an accurate estimate of the error rate with a negligible amount of computation time.

* Only truncated pulse train approximations are considered.

4.1 *Upper and Lower Bound of P_e*

Let the intersymbol interference be partitioned into two disjoint sets where

$$X_N = \sum_{\substack{l \neq 0 \\ l \in S_N}} a_l r(t_0 - lT), \tag{15a}$$

and

$$X_R = \sum_{l \notin S_N} a_l r(t_0 - lT). \tag{15b}$$

Equation (14) can be rewritten as

$$P_e = [(2m - 1)/m] \int_{\text{all } X_N} \int_{\text{all } X_R} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \int_{-\infty}^{0} \exp \{-[y - r(t_0) + X_N + X_R]^2/2\sigma^2\} \, dy \, dF(X_N) \, dF(X_R). \tag{16}$$

Proposition 1: P_e is lower bounded by

$$P_L = [(2m - 1)/m] \int_{\text{all } X_N} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \int_{-\infty}^{0} \exp [-\{y - r(t_0) + X_N\}^2/2\sigma^2] \, dy \, dF(X_N), \tag{17}$$

provided the truncated system has an "open eye pattern," i.e.,

$$r(t_0) - \sum_{\substack{l \in S_N \\ l \neq 0}} | r(t_0 - lT) | \geqq 0. \tag{18}$$

Proof: The complementary error function is concave upwards for negative values of its argument and satisfies the following relationship:

$$\tfrac{1}{2} \operatorname{erfc}(z + \alpha) + \tfrac{1}{2} \operatorname{erfc}(z - \alpha) \geqq \operatorname{erfc}(z), \quad z \leqq 0. \tag{19}$$

Since X_R is symmetrically distributed around zero and X_N satisfies equation (18), we obtain, by applying equation (19), that

$$\int_{\text{all } X_R} \int_{-\infty}^{0} \exp \{-[y - r(t_0) + X_N + X_R]^2/2\sigma^2\} \, dy \, dF(X_R) \geqq \int_{-\infty}^{0} \exp \{-[y - r(t_0) + X_N]^2/2\sigma^2\} \, dy. \tag{20}$$

Substituting equation (20) into equation (16), we obtain the lower bound of equation (17).

Proposition 2: P_e is upper bounded by

$$P_u = [(2m - 1)/m] \int_{\text{all } X_N} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \int_{-\infty}^{0} \exp\{-[y - r(t_0) + X_N]^2/2\sigma_1^2\}\, dy\, dF(X_N), \tag{21a}$$

where

$$\sigma_1^2 = \sigma^2(1 - \sigma_R^2/\sigma^2)^{-1}, \tag{21b}$$

$$\sigma_R^2 = (1/3)(2m - 1)(2m + 1)\sigma_r^2, \tag{21c}$$

and σ_r^2 is defined in equation (12).

Proof: Applying the following inequality,

$$\exp\{-X_R^2/2\sigma^2\} \leqq 1, \tag{22}$$

to equation (16), we obtain

$$P_e \leqq [(2m - 1)/m] \int_{\text{all } X_N} \int_{-\infty}^{0} \int_{\text{all } X_R} (2\pi\sigma^2)^{-\frac{1}{2}} \cdot \exp\{-[y - r(t_0) + X_N]^2/2\sigma^2\} \cdot \exp\{-[y - r(t_0) + X_N]X_R/\sigma^2\}\, dF(X_R)\, dy\, dF(X_N). \tag{23}$$

Knowing from equation (15b),

$$X_R = \sum_{l \notin S_N} a_l r(t_0 - lT),$$

the average over X_R can be performed, we thus have

$$\int_{\text{all } X_R} \exp\{-[y - r(t_0) + X_N]X_R/\sigma^2\}\, dF(X_R) = \prod_{l \notin S_N} \langle \exp\{-[y - r(t_0) + X_N]a_l r(t_0 - lT)/\sigma^2\}\rangle_{a_l}, \tag{24a}$$

where $\langle g(x)\rangle_x$ means expectation of $g(x)$. It has been shown* that the following inequality holds.

$$\langle \exp\{a_l x\}\rangle_{a_l} \leqq \exp(x^2\sigma_{a_l}^2/2) = \exp\{x^2(2m - 1)(2m + 1)/6\}. \tag{24b}$$

Substituting equation (24b) into (24a) we obtain

$$\int_{\text{all } X_R} \exp\{-[y - r(t_0) + X_N]X_R/\sigma^2\}\, dF(X_R) \leqq \exp\{[y - r(t_0) + X_N]^2\sigma_R^2/2\sigma^4\}, \tag{24c}$$

* Appendix of Ref. 6.

where σ_R^2 is given by equation (21c). Substituting equation (24c) into (23) we obtain the upper bound of equation (21a).

It is interesting to note that the upper bound differs from the lower bound only through a modification of the noise power by the truncated terms. For a system with a peak distortion* less than unity, by taking the set S_N large enough σ_R^2 approaches zero, σ_1^2 approaches σ^2, and the upper bound converges to the lower bound. Therefore, the exact error probability can be located within a small range. The computation time involved for large enough N is rather minimal when a digital computer is used as will be illustrated in Section V.

4.2 *Evaluation of P_L and P_u*

We have already shown in Ref. 6 that equations (17) and (21) can be expanded into an absolutely convergent series involving moments of the truncated intersymbol interference.

The series expansion of equation (17) is

$$\begin{aligned} P_L = {} & [(2m-1)/m] \operatorname{erfc}[-r(t_0)/(2^{\frac{1}{2}}\sigma)] \\ & + [(2m-1)/m] \sum_{k=1}^{\infty} [(2k)!]^{-1}(2\sigma^2)^{-k}(\pi)^{-\frac{1}{2}} \exp(-r^2(t_0)/2\sigma^2) \\ & \cdot H_{2k-1}(r(t_0)/(2^{\frac{1}{2}}\sigma))M_{2k}, \end{aligned} \tag{25}$$

where

H_{2k-1} is the Hermite polynomial,
M_{2k} is the $2k$th moment of the random variable X_N.

The series expansion of equation (21) is similar to equation (17),

$$\begin{aligned} P_U = (\sigma_1/\sigma)\Big\{ & [(2m-1)/2m] \operatorname{erfc}[-r(t_0)/(2^{\frac{1}{2}}\sigma_1)] \\ & + [(2m-1)/m] \sum_{k=1}^{\infty} [(2k)!]^{-1}(2\sigma_1^2)^{-k}(\pi)^{-\frac{1}{2}} \exp[-r^2(t_0)/2\sigma_1^2] \\ & \cdot H_{2k-1}(r(t_0)/(2^{\frac{1}{2}}\sigma_1))M_{2k}\Big\}. \end{aligned} \tag{26}$$

The moments (M_{2k}) can be obtained through the characteristic

* Normalized peak distortion (D_p) is defined as

$$D_p = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |r(t_0+nT)|/|r(t_0)|.$$

function of X_N without the explicit evaluation of the distribution function. The recurrence formula for M_{2k} is

$$M_{2k} = -\sum_{i=1}^{k} \binom{2k-1}{2i-1} M_{2(k-i)} (-1)^i \{2^{2i}[(2m)^{2i} - 1]/2i\} \mid B_{2i} \mid \cdot [\sum_{\substack{l \neq 0 \\ l \in S_N}} r(t_0 - lT)^{2i}], \tag{27}$$

where B_{2i} are the Beruoulli numbers.

4.3 *Truncation Error Bound of Series Expansion*

The error incurred by truncating the series of equation (25) at $(n - 1)$ term is given by

$$R_n = [(2m - 1)/m] \sum_{k=n}^{\infty} (2k!)^{-1}(2\sigma^2)^{-k}(\pi)^{-\frac{1}{2}} \exp [r^2(t_0)/2\sigma^2] \cdot H_{2k-1}[r(t_0)/(2^{\frac{1}{2}}\sigma)] \cdot M_{2k} . \tag{28}$$

Let

$$\lambda = \max \mid X_N \mid = (2m - 1) \sum_{\substack{l \in S_N \\ l \neq 0}} \mid r(t_0 - lT) \mid . \tag{29}$$

It can be shown that the moments satisfy

$$M_{2k+2p} \leqq M_{2k}\lambda^{2p}, \quad p = 0, 1, 2, \cdots . \tag{30}$$

For $(2k - 1) \gg x$, the Hermite polynomials are upper bounded by

$$\mid H_{2k-1}(x) \mid \leqq 2^{k-\frac{1}{2}}[(2k - 3)!!] \sqrt{2k - 1} \exp [x^2/2]. \tag{31}$$

Substituting equations (30) and (31) into equation (28) we obtain the following:

$$\begin{aligned} \mid R_n \mid &\leqq [(2m - 1)/m](2\pi)^{-\frac{1}{2}} \exp [-r^2(t_0)/4\sigma^2] \\ &\quad \cdot M_{2n} \cdot (2\sigma^2)^{-n} \sum_{k=n}^{\infty} (k!)^{-1}(2k - 1)^{-\frac{1}{2}}(\lambda^2/2\sigma^2)^{k-n} \\ &\leqq [(2m - 1)/m](2\pi)^{-\frac{1}{2}} \exp [-r^2(t_0)/4\sigma^2] \\ &\quad \cdot M_{2n} \cdot (2\sigma^2)^{-n} \left\{ \sum_{k=n}^{p-1} (k!)^{-1}(2k - 1)^{-\frac{1}{2}}(\lambda^2/2\sigma^2)^{k-n} \right. \\ &\quad \left. + (P!)^{-1}(2p - 1)^{-\frac{1}{2}}(\lambda^2/2\sigma^2)^{p-n}(1 - \lambda^2/2p\sigma^2)^{-1} \right\}, \end{aligned} \tag{32}$$

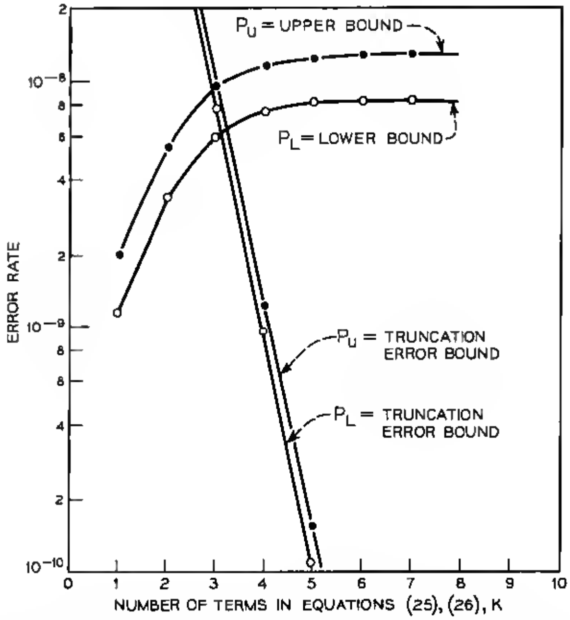

Fig. 2—The convergence of the series expansion method; $(\sin \pi t/T)/(\pi t/T)$ pulse, binary AM system, $t_0 = 0.05\ T$, SNR = 16 dB. The set S_N includes the first 12 sampling points around t_0.

where p is an integer which is chosen to satisfy $(\lambda^2/2p\sigma^2) < 1$. Similar truncation error bounds can be obtained for P_u .

V. APPLICATION

The error probability of a $2M$-ary digital AM system with an ideal band-limiting pulse signal operating over an ideal channel is calculated by equations (25) and (26) to determine the convergence of the method. The received binary pulse is assumed to be

$$r(t) = (\sin \pi t/T)/(\pi t/T). \tag{33}$$

The system SNR is defined by

$$\text{SNR} = \langle a_0^2 \rangle r^2(0)/\sigma^2. \tag{34}$$

The convergence of the series expansion method is illustrated in Fig. 2. The system is binary with the sampling instant deviated by $0.05T$ from its nominal sampling instant. The SNR is 16 dB. The set S_N includes 12 elements, i.e., $l = \pm 1, \pm 2, \cdots, \pm 6$. It is observed that

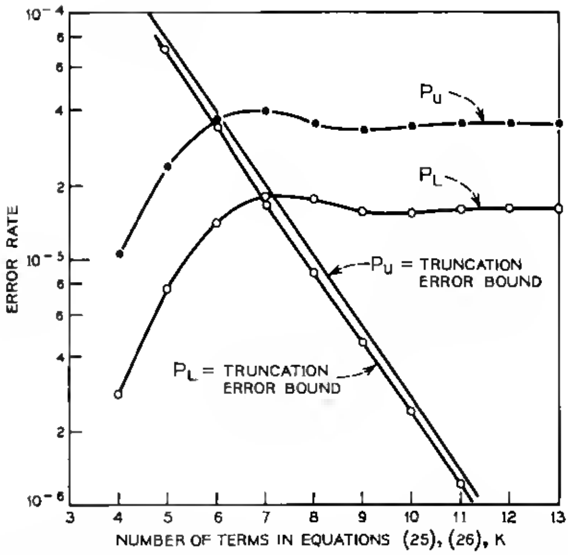

Fig. 3—Convergence of P_L as a function of number of terms in the series expansion; $(\sin \pi t/T)/(\pi t/T)$ pulse, 4-level digital AM system, $t_0 = 0.05\ T$, SNR = 23 dB. The set S_N includes the first 12 sampling points around t_0.

the series converges after 3 or 4 terms. A similar example is given in Fig. 3 for a 4-level system with SNR = 23 dB.

The convergence of the upper bound to the lower bound with increased size of S_N is illustrated in Figs. 4, 5, and 6 for binary and 4-level systems respectively.* It is observed that the two bounds indeed merge together as N is increased. The upper and lower bounds on the error rate were calculated using a program written for the GE-Mark II time-sharing system. For the examples given here, computation time was less than a second. The change of N from 6 to 30 hardly had any effect on the computation time which indicates that one should start with S_N sufficiently large such that σ_R^2 is small in comparison with σ^2, probably of the order of 0.2 σ^2 or smaller. Under this condition the upper and lower bound should be fairly close. As a comparison, the Chernoff bounds are also presented in Figs. 4, 5, and 6.

The method given here can also be applied to the calculation of the error rate of a coherent phase-shift-keyed system. The error rate calculation of a two and four phase system can be reduced to the basic

* The number of terms in the series expansion used to calculate the points in these figures are determined so that the truncation error is negligible.

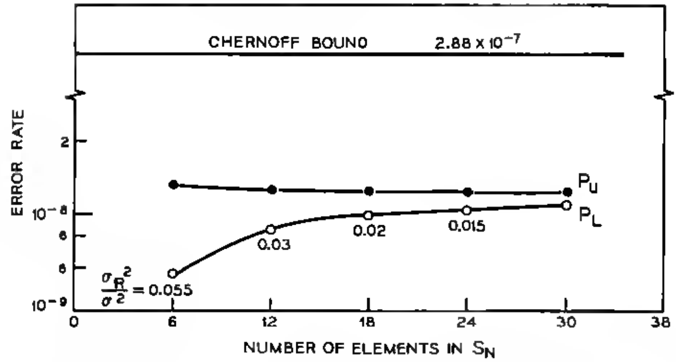

Fig. 4—Convergence of P_u to P_L; ideal band-limited pulse, binary system, SNR = 16 dB, $t_0 = 0.05\ T$.

formula, equation (10),[9] which then requires the determination of

$$P\{\sum_{l \neq 0} a_l r(t_0 - lT) + n(t_0) \geqq r(t_0)\}. \tag{35}$$

The method described here can then be applied. Similar applications can also be found in the error calculation of co-channel interference.

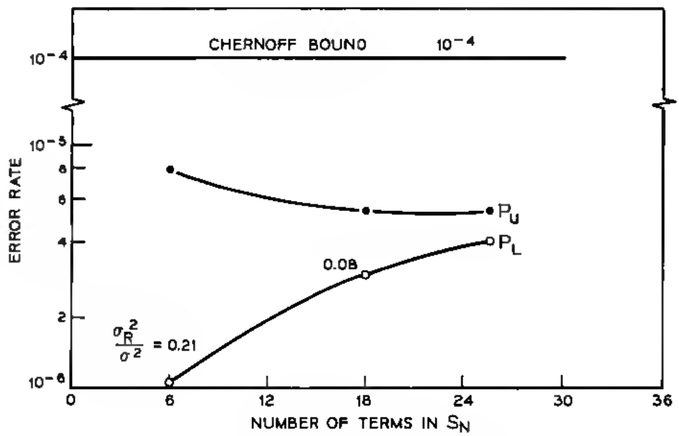

Fig. 5—Convergence of P_u to P_L; ideal band-limited pulse, binary system, SNR = 16 dB, $t_0 = 0.1\ T$.

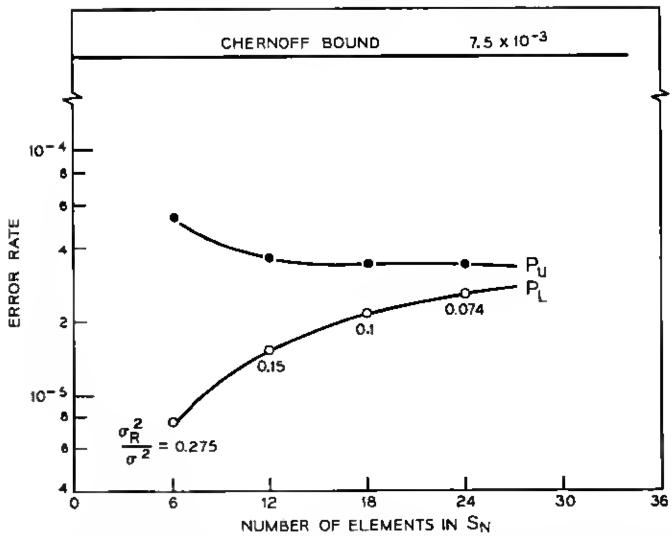

Fig. 6.—Convergence of P_u to P_L; ideal band-limited pulse, 4-level system, SNR = 16 dB, $t_0 = 0.05\ T$.

VI. CONCLUSIONS

We have presented a method to calculate the error rate of a coherent digital system subject to intersymbol interference and additive Gaussian noise. The error rate for a system with a peak distortion less than unity can be determined to arbitrary accuracy through the calculation of an upper bound and a lower bound of the error rate. The computation time involved (less than one second on the GE-Mark II time-sharing system) is many orders of magnitude shorter than the time required by the straightforward calculation of all the possible states. On the other hand, the results are generally much more accurate than the results obtained through the application of the Chernoff bound.

VII. ACKNOWLEDGEMENT

The authors wish to thank J. Salz and D. A. Spaulding for many helpful discussions.

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